Q.2 a. Evaluate $\underset{x\to 0}{LT}\dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$

Answer

Here we have, $\underset{x\to 0}{Lt}\dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$

$$= \underset{x\to 0}{Lt}\frac{2\sin\left(\frac{2x+6x}{2}\right)+\cos\left(\frac{2x-6x}{2}\right)}{2\cos\left(\frac{5x+3x}{2}\right).\sin\left(\frac{5x-3x}{2}\right)}$$

$$= \underset{x\to 0}{Lt}\frac{\sin 4x\cos(-2x)}{\cos 4x.\cos x}$$

$$= \underset{x\to 0}{Lt}4\left(\frac{\sin 4x}{4x}\right)\left(\frac{x}{\sin x}\right)\left(\frac{\cos 2x}{\cos 4x}\right)$$

$$= 4(1)\,(1)\,(1) = 4$$

b. If f is a real function defined by $f(x)=\dfrac{x-1}{x+1}$ then prove that $f(2x)=\dfrac{3f(x)+1}{f(x)+3}$

Answer

$f(x) = \dfrac{x-1}{x+1}$, f(x) + 1

$$= \frac{x-1}{x+1}+1 = \frac{(x-1)+(x+2)}{(x+1)}$$

$$f(x) -1 = \frac{x-1}{x+1}-1=\frac{x-1+x+1}{x+1}$$

$$\Rightarrow \text{then}\frac{f(x)+1}{f(x)-1}=\frac{x-1+x+1}{x+1}$$

(Applying componendo & devidendo)

$\Rightarrow x=\dfrac{f(x)+1}{1-f(x)}$, Now, $f(2x) = \dfrac{2x-1}{2x+1}$

$$\Rightarrow f(2x) = \frac{\left\{\frac{f(x)+1}{1-f(x)}\right\} - 1}{2\left\{\frac{f(x)+1}{1-f(x)}\right\} + 1}$$

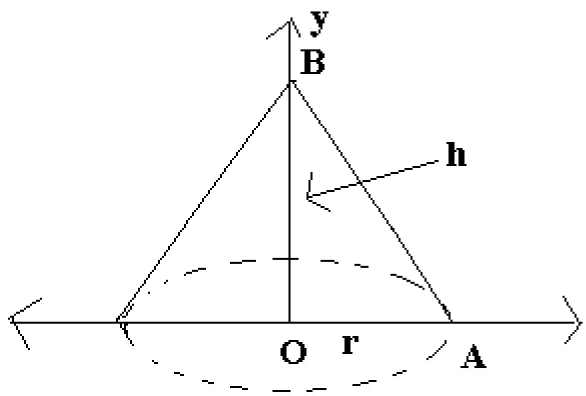
$$= \frac{2f(x)+2-1+f(x)}{2f(x)+2+1-f(x)}$$

$$\Rightarrow f(2x) = \frac{3f(x)+1}{f(x)+3}$$ There proof

Q.3 a. Find the volume of the right circular cone formed by the revolution of a right angled triangle about a side which contains the right angle.

Answer

Let O B A be the right angled with OA = r and OB = h. When the triangle is revolved about the side y-axis is about the side OB. We get a right circular cone of radius r and height h.



The curve is the line AB, whose eqn. is

$$\frac{x}{r} + \frac{y}{h} = 1$$

or $x = \frac{r}{h}(h-y)$ _______________ (i)

Required volume = $\pi \int_0^{\pi} \frac{r^2}{h^2}(h-y)^2 dy$

$$= \frac{\pi r^2}{h^2}\left[-\frac{(h-y)^3}{3}\right]^h$$

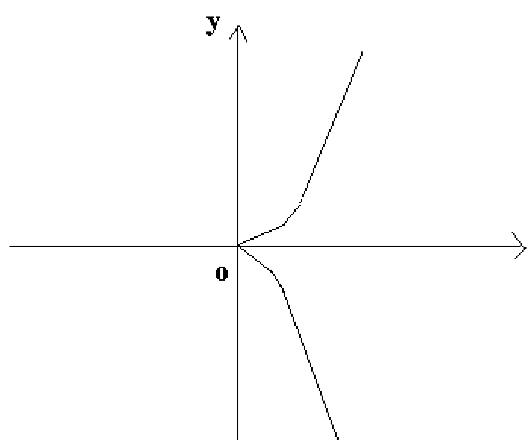
$$= \frac{\pi r^2}{h^2}\left[0+\frac{h^3}{3}\right]^0 = \frac{1}{3}\pi r^2 h$$

b. Find the length of the curve $y^2 = x^3$ from origin to the point (1, 1).

Answer

The curve can easily be traced and its shape is shown in below figure

The eqn. of the curve is $y^2 = x^3$ _____________ (i)



$y^2 = x^3$

$$\therefore 2y\frac{dy}{dx} = 3x^2$$

or $$\frac{dy}{dx} = \frac{3x^2}{2y} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}\,[\text{from(i)}]$$

Now, $$S = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{\left(1+\frac{9x}{4}\right)}dx$$

$$= \frac{1}{2}\int_0^1 \sqrt{4+9x}.dx$$

$$= \frac{1}{2}\left[\frac{1}{9}\times\frac{2}{3}(4+9x)^{3/2}\right]_0^1$$

$$= \frac{1}{27}\left[(13)^{3/2} - (4)^{3/2}\right]$$

$$= \frac{1}{27}\left[13\sqrt{13} - 8\right]$$

Q.4 a. If n is a positive integer then show that $\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = 2^{n+1}\cos\frac{n\pi}{6}$ where i = $\sqrt{-1}$

Answer

Let

$$\sqrt{3} + i = r(\cos\alpha + i\sin\alpha)$$

$$r = \sqrt{3+1=2}, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\sqrt{3} + i = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$(\sqrt{3}+i)^n + \left(\sqrt{3}-i\right)^n = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^n + \left[2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^\pi$$

$$= \quad 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) + 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

$$= 2^{n+1}\cos\frac{n\pi}{6}$$ R.H.S. Hence proved

b. A resistance of 20 ohms and inductance of 0.2 H and a capacitance of 100 μF are connected in series a cross 220 Volt, 50cycle/sec main. Determine: (i) impedance (ii) current (iii) voltage across L,R and C (iv) power in watt (v) power factor

Answer There, R = 20Ω
L = 0.2 H
C = 100μf
V = 200V
f = 50 c/s

(a) Impendence (z) = R – J × C + J × L

$$= 20 - j\ \frac{1}{wc} + j\ Lw$$

$$= 20 - j\,\frac{1}{50\times 2\pi\times 100\times 10^{-6}} + j(0.2)2\pi\times 50$$

$$= 20 - J\,\frac{100}{\pi} + J20\pi = 20 - j\,31.831 + J\,62.8319$$

$$= 20 + J\,31$$

So. $|Z| = \sqrt{400+961} = \sqrt{1361}$

$|Z|$ = 36.89 ohms.

(b) $i = \frac{v}{|z|} = \frac{200}{36.89} = 5.42$

(c) $V_L = i \times L$

$= 5.42 \times 2\,\pi\, f \times 0.2$

$= 5.42 \times 2 \times \pi \times 50 \times 0.2$

$V_R = iR = 5.42 \times 20 = 108.4$ volts

$$V_c = i \times \frac{1}{2\pi f \times c} = \frac{5.42\times 1}{2\times\pi\times 50\times 100\times 10^{-6}} = 172.52\text{volts}$$

(d) Power = i^2R

$= (5.42)^2 \times 20$ 587.528 watts.

(e) Power factor $= \frac{R}{|z|} = \frac{20}{36.89} = 0.542$

Q.5 a. A rigid body is spinning with an angular velocity of 27 radian/second about an axis parallel to 2i +j–2k passing through the point i+3j-k. Find the velocity of the point whose position vector is 4i+8j+k.

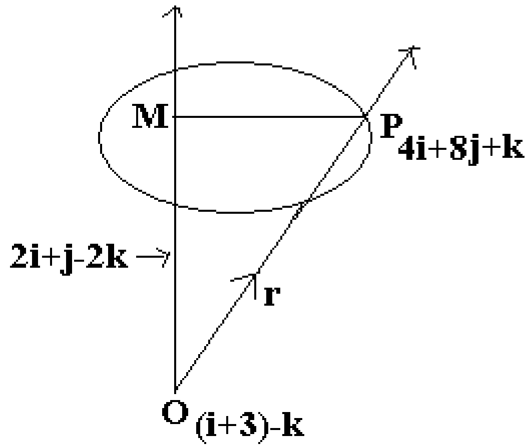
Answer

Let $\vec{w}$ be the angular of the body rotating about an axis parallel to the vector 2i + j – 2k.

Then $\vec{w} = 27 \times \frac{2i + j - 2k}{\sqrt{2^2 + i^2 + (-2)^2}}$

$\vec{w}$ = 18i + 9j – 18k

Let $\vec{r} = \overrightarrow{OP}$ = P.V. of $\vec{P}$ P.V. of $\vec{O}$



$\vec{r}$ = (4i +8j+k)-(i+3j-k)

$\vec{r}$ = 3i +5j +2k

$$\text{Let } \vec{u} = \vec{w} \times \vec{r} = \begin{vmatrix} i & j & k \\ 18 & 9 & -18 \\ 3 & 5 & 2 \end{vmatrix}$$

$\Rightarrow \vec{u}$ = 108i – 90j + 63k

= 9(12i – 10j +7k)

b. Find the area of the triangle formed by the point whose position vectors are 3i+j, 5i+2j+k, i-2j+3k.

Answer Let ABC be a triangle and let $\vec{a} = 3i + j$, $\vec{b} = 5i + 2j + k$, and $\vec{c} = i - 2j + 3k$ be the position vectors of its vertices A, B, and C respectively, then

$\overrightarrow{AB}$ = P.V. of B – P.V. of A

= (5i +2j+k) – (3i+j) = 2i + j + k

and $\overrightarrow{AC}$ = P.V. of B – P.V. of A

= (i-2j+3k) – (3i + j) = -2i -3j + 3k

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -2 & -3 & 3 \end{vmatrix} = 6i - 8j - 4k$$

$$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{36+64+16} = \sqrt{116}$$

$$\therefore \text{ Reqd. Area } = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{116} = \sqrt{29}$$

(formula, Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BA}\right| = \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$

Q.6 a. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

Answer

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

$(D^2 + 6D + 9)\, y = 5e^{3x}$

Auxiliary equation is $D^2 + 6D + 9 = 0$ or $D = -3, -3$,

C.F. = $(C_1 + C_2 x)e^{-3x}$

$$= \frac{1}{D^2 + 6D + 9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2 + 6(3) + 9}$$

$$= \frac{5e^{3x}}{36}$$

b. Solve $\frac{d^2y}{dx^2} + 9y = \sec 3x$

Answer

$$\frac{d^2y}{dx^2} + 9y = \sec 3x$$

Auxiliary equation $D^2 + 9 = 0$ or $D = \pm 3i$

C.F = $C_1 \cos 3x + C_2 \sin 3x$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right].\sec 3x \quad \underline{\hspace{3em}} \ (1)$$

Now $\frac{1}{D-3i}\sec 3x = e^{3ix}\int e^{-3ix} \sec 3x\, dx$

$$= e^{3ix}\int \frac{\cos 3x - i\sin 3x}{\cos 3x}dx = e^{3ix}\int (1 - i\tan 3x)dx$$

$= e^{3ix}\left(x+\frac{i}{3}\ell og\cos 3x\right)$

Changing I to –I, we have

$$\frac{1}{D+3i}\sec 3x = e^{-3ix}\left(x-\frac{i}{3}\ell og\cos 3x\right)$$

Putting these values in (i), we get

$$\text{P.I.} = \frac{1}{6i}\left[e^{3ix}\left(x+\frac{i}{3}\ell og\cos 3x\right)-e^{-3ix}\left(x-\frac{i}{3}\log\cos 3x\right)\right]$$

$$= \frac{x}{6i}e^{3ix}+\frac{\ell og\cos 3x}{18}-\frac{x}{6i}e^{-3ix}+\frac{e^{-3ix}}{18}\ell og\cos 3x$$

$$= \frac{x}{3}e^{3ix}+\frac{e^{3ix}\ell og\cos 3x}{18}-\frac{xe^{-3ix}}{6i}+\frac{e^{-3ix}}{18}\ell og\cos 3x$$

$$= \frac{x}{3}\frac{e^{3ix}-e^{-3ix}}{2i}+\frac{1}{9}.\frac{e^{3ix}+e^{-3ix}}{2}\ell og\cos 3x$$

$$=\frac{x}{3}.\frac{e^{3ix}-e^{3ix}}{2i}+\frac{1}{9}.\frac{e^{3ix}+e^{-3ix}}{2}\ell og\cos 3x$$

$$=\frac{x}{3}\sin 3x+\frac{1}{9}.\cos 3x.\ell og\cos 3x$$

Hence, complete solution is $y = C_1\cos 3x + \frac{x}{3}\sin 3x+\frac{1}{9}.\cos 3x.\ell og\cos 3x$

Q.7 a. Expand f(x) = e^{x} in a cosine series over (0, 1)

Answer

Here

F(x) = x and C =1

$$\therefore a_0 = \frac{2}{c}\int_0^c f(x)dx = \frac{2}{1}\int_0^1 e^x dx = 2(e-1)$$

$$a_n = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x+\cos n\pi x\right]_0^1$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi + \cos n\pi) - \frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}\left[(-1)^n e - 1\right]$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + ...$$

$$e^x = e-1 + 2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + ...\right]$$

b. Find the Fourier Series of the function

$$f(t) = \begin{cases} 0 & \text{when} \quad -2 < t < -1 \\ K & \text{"} \quad -1 < t < 1 \\ 0 & \text{"} \quad 1 < t < 2 \end{cases}$$

Answer

2C = 4 or C = 2

$$a_0 = \frac{1}{C}\int_{-c}^{c} f(t)dt$$

$$= \frac{1}{2}\int_{-c}^{c} k dt = \frac{k}{2}(t)_{-1}^{t} = \frac{k}{2}(1+1) = k$$

$$an = \frac{1}{C}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{C}dt$$

$$= \frac{1}{2}\int_{-c}^{c} k\cos\frac{n\pi t}{C}dt$$

$$= \frac{k}{2}\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right)_{-1}^{t}$$

$$= \frac{k}{n\pi}\left(\sin\frac{n\pi}{2} - \sin-\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{C}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{2}dt$$

$$= \frac{1}{2}\int_{-c}^{c} k\sin\frac{n\pi t}{2}\,dt$$

$$= \frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_{-1}^{1} \quad = \frac{-k}{n\pi}\left(\cos\frac{n\pi}{2}-\cos\frac{n\pi}{2}\right) = 0$$

Fourier series is

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{c} + a_2\cos\frac{2\pi t}{c} + a_3\cos\frac{3\pi t}{c} + \ldots\ldots\ldots\ldots$$

$$+\ b_1\sin\frac{\pi t}{c} + b_2\sin\frac{2\pi t}{c} + b_3\sin\frac{3\pi t}{c} + \ldots\ldots\ldots\ldots$$

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\sin\frac{\pi}{2}.\cos\frac{\pi t}{2} + \frac{1}{2}\sin\pi.\cos\frac{2\pi t}{2} + \frac{1}{3}\sin\frac{3\pi}{2}.\cos\frac{3\pi t}{2} + \ldots.\right]$$

$$= \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \ldots\ldots..\right]$$

Q.8 a. Evaluate $L\{te^{-t}\cosh t\}$

Answer

$$L\{e^{-t}\cosh t\} = L\left\{e^{-t}\left(\frac{e^{t}+e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{1+e^{-2t}\}$$

$$= \frac{1}{2}\left(\frac{1}{s}+\frac{1}{s+2}\right)$$

$$\therefore (te^{-t}\cosh t) = -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s}+\frac{1}{s+2}\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{2}\frac{(s+2)^2+s^2}{s^2(s+2)^2}$$

$$= \frac{s^2+2s+2}{s^2(s+2)^2}$$

b. Evaluate $L\left\{\int_0^t \frac{e^t \sin t}{t} dt\right\}$

Answer

Here to find $L\left[\int_0^t \frac{e^t \sin t}{t} dt\right]$

We have

$L(\sin t) = \frac{1}{s^2+1}$

$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}$

$= \left[\tan^{-1} s\right]_0^\infty$

$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

$\therefore L\left(e^t \frac{\sin t}{t}\right) = \cot^{-1}(s-1) = f(s)$

$\therefore L\left[\int_0^t e^t \frac{\sin t}{t} dt\right] = \frac{1}{s} f(s)$

$= \frac{1}{s}\cot^{-1}(s-1)$

Q.9 a. Show that $L^{-1}\left\{\frac{S^2}{S^4+4a^4}\right\} = \frac{1}{2a}(\cosh at.\sin at + \sinh at.\cos at)$

Answer

$L^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}$

$= L^{-1}\left\{\frac{s^2}{(s^4+2a^2)^2 - 4a^2s^2}\right\}$

$= L^{-1}\left\{\frac{s^2}{(s^4+2a^2)^2 - 4a^2s^2}\right\}$

$= L^{-1}\left\{\frac{1}{4a}\left(\frac{s}{s^4+2a^2-4a^2s^2}\right) - \frac{s}{(s^4+2a^2+2as)}\right\}$

Resolving into partial fraction.

$$= \frac{1}{4a}\left[L^{-1}\left\{\frac{s}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{(s+a)-a}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[L^{-1}\left\{\frac{(s-a)+a}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{(s+a)-a}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}.L^{-1}\left\{\frac{s+a}{s^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{(s-a0}{s^2+a^2}\right\}\right]$$

by first shifting theorem.

$$= \frac{1}{4a}\left[e^{at}.L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - ae^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at}.L^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\left(e^{at}-e^{-at}\right).L^{-1}\left\{\frac{1}{s^2+a^2}\right\} + a\left(e^{at}-e^{-at}\right).L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[(2\sinh at)\cos at + a(2\cosh at)\left(\frac{1}{a}\right).\sin at\right]$$

$$= \frac{1}{2a}\left[\cosh at.\sin at + \sinh at.\cos at\right] \quad \text{Hence proved}$$

b. Evaluate $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Answer

To evaluate

$$L^{-1}\left\{\ell og\frac{s+1}{s-1}\right\}$$

Let $L^{-1}\left\{\ell og\frac{s+1}{s-1}\right\} = f(t)$

Then $L\{f(t)\} = f(s) = \ell og\frac{s+1}{s-1}$

$$\therefore L\{t+(t)\} = (-1)\frac{d}{ds}\left\{\ell og\frac{s+1}{s-1}\right\}$$

$$= -\frac{d}{ds}\{\ell og(s+1) - \ell og(s-1)\}$$

$= - = -\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$= \; = \frac{1}{s-1} - \frac{1}{s+1}$

$= \; L\left\{e^{t} - e^{-t}\right\}$

$= \; L\left\{2\sinh t\right\}$

$\therefore tf(t) = 2\sinh t$

or $f(t) = \frac{2\sinh t}{t}$

Hence $L^{-1}\left\{\ell og \frac{s+1}{s-1}\right\} = \frac{2\sinh t}{t}$

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